

A THIN STRINGY MODULI SPACE FOR SLODOWY SLICES

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ABSTRACT. The paper provides new examples of an explicit submanifold in Bridgeland stabilities space of a local Calabi-Yau.

More precisely, let X be the standard resolution of a transversal slice to an adjoint nilpotent orbit of a simple Lie algebra over \mathbb{C} . An action of the affine braid group on the derived category $D^b(\text{Coh}(X))$ and a collection of t -structures on this category permuted by the action have been constructed in [BR] and [BM] respectively. In this note we show that the t -structures come from points in a certain connected submanifold in the space of Bridgeland stability conditions. The submanifold is a covering of a submanifold in the dual space to the Grothendieck group, and the affine braid group acts by deck transformations. In the special case when $\dim(X) = 2$ a similar (in fact, stronger) result was obtained in [Br1]. The dimension of our subset equals (in most cases) that of the second cohomology of X , so it may deserve the name of stringy moduli space; it is in a sense smaller than one may want, hence the attribute "thin".

We also propose a new variant of definition of stabilities on a triangulated category, which we call a "real variation of stability conditions" and discuss its relation to Bridgeland's definition. The main theorem provides an illustration of such a relation.

1. INTRODUCTION AND STATEMENT OF RESULT

Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} , let $e \in \mathfrak{g}$ be a nilpotent element, $S \subset \mathfrak{g}$ be a transversal (Slodowy) slice to the G -orbit of e . Let $\mathcal{B} = G/B$ be the flag variety of G , and $\pi : T^*(\mathcal{B}) \rightarrow \mathfrak{g}$ be the Springer (moment) map. Set $\mathcal{B}_e = \pi^{-1}(e)$ and $X = \pi^{-1}(S)$. Set $\mathcal{C} = D^b(\text{Coh}_{\mathcal{B}_e}(X))$ where $\text{Coh}_{\mathcal{B}_e}(X)$ is the category of coherent sheaves on X supported on \mathcal{B}_e .

Certain t -structures on \mathcal{C} were constructed in [BM] (announced in [B]). In this note we show that they arise from a certain explicit connected subset of the space $\text{Stab}(\mathcal{C})$ of Bridgeland stability conditions on \mathcal{C} . To state the result we need more notations.

Let \mathfrak{h} denote the (abstract) Cartan algebra of \mathfrak{g} .

We have $\mathfrak{h}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ where Λ is the weight lattice. Let $\mathfrak{h}_{\mathbb{R}}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{h}^*$ be the real dual Cartan. The affine Weyl group W_{aff} acts on \mathfrak{h}^* and on $\mathfrak{h}_{\mathbb{R}}^*$ by affine-linear transformations. Let $\mathfrak{h}_{\text{reg}}^*$ be the union of free orbits of W_{aff} on \mathfrak{h}^* , thus $\mathfrak{h}_{\text{reg}}^*$ is the complement to the affine coroot hyperplanes $H_{\check{\alpha}, n} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle = n\}$, where $n \in \mathbb{Z}$ and $\check{\alpha}$ is a coroot.

1.1. Action of the affine braid group and t -structures assigned to alcoves.

Recall that an *alcove* is a connected component of $\mathfrak{h}_{\mathbb{R}, \text{reg}}^* = \mathfrak{h}_{\text{reg}}^* \cap \mathfrak{h}_{\mathbb{R}}^*$. The natural action of the affine Weyl group on \mathfrak{h}^* induces a simply transitive action of W_{aff} on the set of alcoves. We denote this set by Alc .

The argument below is based on the construction of [BM] which assigns a t -structure τ_A on \mathcal{C} to an alcove $A \in Alc$. The t -structure τ_A can be described using derived localization over a field of characteristic $p > 0$ [BMR1]. Roughly speaking, modules for the sheaf $D_\lambda(\mathcal{B})$ of twisted differential operators on \mathcal{B} are closely related to coherent sheaves on $T^*(\mathcal{B})$; on the other hand, the derived category $D^b(D_\lambda(\mathcal{B}))$ is identified with the derived category of an appropriate quotient of the enveloping algebra $U\mathfrak{g}$. Thus one can get a t -structure on $D^b(Coh(T^*(\mathcal{B})))$ which is compatible with the tautological t -structure on $D^b(U\mathfrak{g} - mod)$. The t -structure τ_A arises this way when the twisting parameter λ satisfies the condition $\frac{\lambda + \rho}{p} \in A$. There exists also a more direct construction of the t -structure τ_A over a characteristic zero field, though available proof of its properties relies on positive characteristic picture.

Let $B_{aff} = \pi_1(\mathfrak{h}_{reg}^*/W_{aff})$ be the affine braid group (this is the affine braid group of Langlands dual group in the standard terminology). An action of B_{aff} on \mathcal{C} was defined in [BR]. This action permutes the t -structures τ_A . More precisely, to each pair of alcoves $A, A' \in Alc$ one can assign an element $b_{A,A'} \in B_{aff}$; it is then shown in [BM] that $b_{A,A'}$ sends τ_A to $\tau_{A'}$. To define $b_{A,A'}$ notice that an element in B_{aff} is determined by a homotopy class of a path connecting two alcoves in \mathfrak{h}_{reg}^* . The element $b_{A,A'}$ corresponds to a path $\phi : [0, 1] \rightarrow \mathfrak{h}_{reg}^*$ such that $\phi(0) \in A$, $\phi(1) \in A'$ and $\phi(t) \in \mathfrak{h}_{\mathbb{R}}^* + i(\mathfrak{h}_{\mathbb{R}}^*)^+$ for $t \in (0, 1)$; here $(\mathfrak{h}_{\mathbb{R}}^*)^+ \subset \mathfrak{h}_{\mathbb{R}}^*$ is the dominant Weyl chamber. This requirement characterizes the homotopy class of ϕ uniquely.

For future reference we fix a universal covering $\widetilde{\mathfrak{h}_{reg}^*}$. We also fix a continuous lifting of each alcove $A \in Alc$ to a subset \widetilde{A} in $\widetilde{\mathfrak{h}_{reg}^*}$, so that for each two alcoves A, A' a path representing $b_{A,A'}$ lifts to a continuous path connecting \widetilde{A} to $\widetilde{A'}$.

1.2. Embedding $\mathfrak{h}^* \rightarrow K^0(\mathcal{C})^*$ and the "quasi-exponential" map. We identify $H^*(G/B, \mathbb{C})$ with $K^0(Coh(G/B)) \otimes \mathbb{C}$ by means of the Chern character map. Notice that the class of the line bundle $\mathcal{O}(\lambda)$ attached to $\lambda \in \Lambda$ corresponds to $\exp(\lambda) \in H^*(G/B)$ where $\lambda \in \Lambda$ is considered as an element in $\mathfrak{h}^* = H^2(G/B)$; it is a nilpotent element in the commutative algebra $H^*(G/B)$, so its exponent is well defined.

We have a bilinear pairing $K^0(G/B) \times K^0(\mathcal{C}) \rightarrow \mathbb{Z}$ given by $([\mathcal{F}], [\mathcal{G}]) = \chi(pr^*(\mathcal{F}) \otimes \mathcal{G})$. Here χ stands for Euler characteristic and pr for the projection $T^*(G/B) \rightarrow G/B$. This gives a map $H^*(G/B) \rightarrow (K^0(\mathcal{C}) \otimes \mathbb{C})^*$. We will omit complexification from notation where it is not likely to lead to a confusion, and identify an element in $H^*(G/B)$ with its image in $K^0(\mathcal{C})^*$.

We extend the map $\Lambda \rightarrow H^*(G/B) \cong K^0(Coh(G/B)) \otimes \mathbb{C}$, $\lambda \mapsto \exp(\lambda)$ to \mathfrak{h}^* as follows. Define the "quasi-exponential" map $E : \mathfrak{h}^* = \mathfrak{h}_{\mathbb{R}}^* \times (\sqrt{-1}\mathfrak{h}_{\mathbb{R}}^*) \rightarrow H^*(G/B)$ by:

$$E : x + \sqrt{-1}y \mapsto \exp(x)(1 + \sqrt{-1}\exp(y)).$$

In fact, the map $x + \sqrt{-1}y \mapsto (x, x + y)$ is a W_{aff} equivariant isomorphism between \mathfrak{h}^* and $(\mathfrak{h}_{\mathbb{R}}^*)^2$, where W_{aff} acts on $(\mathfrak{h}_{\mathbb{R}}^*)^2$ diagonally. Written as a map from $(\mathfrak{h}_{\mathbb{R}}^*)^2$, the map E takes the form $(\lambda, \mu) \mapsto \exp(\lambda) + \sqrt{-1}\exp(\mu)$.

Remark 1. A variation of the argument below also works for the map $E(z) = \exp(z)$ (with a less explicit and not necessarily open, though still connected neighborhood V of $(\mathfrak{h}_{\mathbb{R}}^*)^{ar}$). The proof of the statement involving the above quasi-exponential map is a bit shorter, so we opted for presenting that version of the result.

Lemma 1. *The map E is compatible with the W_{aff} action where the action on the source is the standard affine linear action on \mathfrak{h}^* , and the one on the target is induced by the B_{aff} action on $D^b(Coh(T^*(\mathcal{B})))$ from [BM].*

Proof. Translations act on the target by twisting with a line bundle and on the source by shifting the real part, thus it is easy to deduce that the map is compatible with the action of the lattice of translations. Compatibility with the action of the finite Weyl group W follows from [BM, Theorem 1.3.2]. \square

1.3. The main result. Define the "almost regular" part $(\mathfrak{h}_{\mathbb{R}}^*)^{ar}$ of the real Cartan as the set of points in $\mathfrak{h}_{\mathbb{R}}^*$ whose stabilizer in W_{aff} has at most two elements. For $\lambda, \mu \in \mathfrak{h}_{\mathbb{R}}^*$ we will write $\lambda \preceq \mu$ if λ lies in the closure of the face which contains μ . Here by a face we mean a stratum of the stratification of $\mathfrak{h}_{\mathbb{R}}^*$ cut out by the coroot hyperplanes (thus alcoves are faces of maximal dimension). Define a neighborhood V of $(\mathfrak{h}_{\mathbb{R}}^*)^{ar}$ in $\mathfrak{h}^* = (\mathfrak{h}_{\mathbb{R}}^*)^2$ by:

$$V = \{(\lambda, \mu) \in (\mathfrak{h}_{\mathbb{R}}^*)^{ar} \times \mathfrak{h}_{\mathbb{R}}^* \mid \lambda \preceq \mu \bigvee \lambda \in (\mathfrak{h}_{\mathbb{R}}^*)^{reg}\}.$$

Thus V is an open neighborhood of $(\mathfrak{h}_{\mathbb{R}}^*)^{ar}$ in \mathfrak{h}^* . Let $V^{reg} = V \cap \mathfrak{h}_{reg}^*$; we have: $V^{reg} = \{(\lambda, \mu) \in (\mathfrak{h}_{\mathbb{R}}^*)^{ar} \times \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda \in (\mathfrak{h}_{\mathbb{R}}^*)^{reg}) \bigvee (\lambda \in \bar{A}, \mu \in A \text{ for some } A \in Alc)\}$. Let $\widetilde{V^{reg}}$ be the preimage of V^{reg} in $\widetilde{\mathfrak{h}_{reg}^*}$.

Theorem 1. *There exists a unique map $\iota : \widetilde{V^{reg}} \rightarrow Stab(\mathcal{C})$ such that*

- 1) *The composed map $Z \circ \iota$, where Z is the projection $Stab \rightarrow K^0(\mathcal{C})^*$, coincides with the map $\sqrt{-1}E \circ \pi$ where π is the projection $\mathfrak{h}_{reg}^* \rightarrow \mathfrak{h}_{reg}^*$.*
- 2) *For some (equivalently, for any) $A \in Alc$ and $z \in \tilde{A}$ the underlying t -structure of the stability $\iota(z)$ coincides with τ_A .*

Proof. Uniqueness of a map ι satisfying (1) and (2) for some fixed alcove A and $z \in \tilde{A}$ follows from a Theorem of Bridgeland [Br] which asserts that the map Z is a local homeomorphism. It remains to show existence of a map ι which satisfies (1) and (2) for all $A, z \in \tilde{A}$. This will be done in section 3.

Example 1. If $\dim(X) = 2$, i.e. e is sub-regular, X is well known to be the minimal resolution of a Kleinian singularity. In this case a component of the space $Stab(\mathcal{C})$ was described in [Br1]. It is easy to see that our submanifold is contained in the one described in *loc. cit.*

Remark 2. It is easy to show that $\pi_1(V^{reg}/W_{aff})$ is a free group with $rank(G)$ generators. This group surjects to $B_{aff} = \pi_1(\mathfrak{h}_{reg}^*/W_{aff})$. (The same remains true if V is replaced by any sufficiently small convex W_{aff} invariant neighborhood of $(\mathfrak{h}_{\mathbb{R}}^*)^{ar}$ in \mathfrak{h}^*). Thus the covering $\widetilde{V^{reg}} \rightarrow V^{reg}$ is connected but is far from being universal. It would be interesting to construct an explicit subset in $Stab(\mathcal{C})$ which is a universal covering of a domain whose fundamental group is isomorphic to the affine braid group. The difficulty seems to come from the fact that for an irreducible object L in the heart of τ_A the corresponding functional d_L can vanish on several faces of A (see below).

1.4. Real variations of stabilities. In this subsection we discuss the motivation for the main result and suggest a new definition.

The real dimension of the manifold $\widetilde{V^{reg}}$ is twice the second Betti number of G/B ; in almost all cases (in particular, in all cases when G is simply-laced except for the

degenerate case when e is regular and X is a point; see [LNS, Theorem 1.3] for the list of exceptional cases) this is equal to twice the second Betti number of X . It is our understanding that physicists expect a canonical submanifold in the stability space $Stab(D^b(Coh(M)))$ for a Calabi-Yau manifold M of real dimension $2b_2(M)$, which they call stringy moduli space (under mirror duality it should correspond to a covering of the moduli space of deformations of the dual Calabi-Yau manifold). We hope that the submanifold $\widetilde{V^{reg}} \subset Stab(\mathcal{C})$ is related to the stringy moduli space of X (hence the title of the note).

More precisely, we conjecture that the following structure is relevant, at least in some examples, for understanding some aspects of the structure of Calabi-Yau categories which have been studied in the literature via the concept of Bridgeland stabilities.

Let \mathcal{C} be a finite type triangulated category and V a real vector space. Suppose that a discrete collection Σ of affine hyperplanes in V is fixed, let V^0 denote their complement. For each hyperplane in Σ consider the parallel hyperplane passing through zero, let Σ_{lin} be the set of those linear hyperplanes. Fix a component V^+ of the complement to the union of hyperplanes in Σ_{lin} . The choice of V^+ determines for each $H \in \Sigma$ the choice of the positive half-space $(V \setminus H)^+ \subset V \setminus H$, where $(V \setminus H)^+ = H + V^+$. By an *alcove* we mean a connected component of the complement to hyperplanes in Σ and we let Alc denote the set of alcoves. For two alcoves $A, A' \in \text{Alc}$ sharing a codimension one face which is contained in a hyperplane $H \in \Sigma$ we will say that A' is *above* A and A is *below* A' if $A' \in (V \setminus H)^+$.

Definition 1. A *real variation of stability conditions* on \mathcal{C} parametrized by V^0 and directed to V^+ is the data (Z, τ) , where Z (the central charge) is a polynomial map $Z : V \rightarrow (K^0(\mathcal{C}) \otimes \mathbb{R})^*$, and τ is a map from Alc to the set of bounded t -structures on \mathcal{C} , subject to the following conditions.

- (1) If M is a nonzero object in the heart of $\tau(A)$, $A \in \text{Alc}$, then $\langle Z(x), [M] \rangle > 0$ for $x \in A$.
- (2) Suppose $A, A' \in \text{Alc}$ share a codimension one face H and A' is above A . Let \mathcal{A} be the heart of $\tau(A)$, and let $\mathcal{A}_n \subset \mathcal{A}$ be the full subcategory in \mathcal{A} given by: $M \in \mathcal{A}_n$ if the polynomial function on V , $x \mapsto \langle Z(x), [M] \rangle$ has zero of order at least n on H . One can check that \mathcal{A}_n is a Serre subcategory in \mathcal{A} , thus $\mathcal{C}_n = \{C \in \mathcal{C} \mid H_{\tau(A)}^i(C) \in \mathcal{A}_n\}$ is a thick subcategory in \mathcal{C} . We require that
 - (a) The t -structure $\tau(A')$ is compatible with the filtration by thick subcategories \mathcal{C}_n .
 - (b) The functor of shift by n sends the t -structure on $gr_n(\mathcal{C}) = \mathcal{C}_n/\mathcal{C}_{n+1}$ induced by $\tau(A)$ to that induced by $\tau(A')$. In other words,

$$gr_n(\mathcal{A}') = gr_n(\mathcal{A})[n]$$

where \mathcal{A}' is the heart of $\tau(A')$, $gr_n = \mathcal{A}'_n/\mathcal{A}'_{n+1}$, $\mathcal{A}'_n = \mathcal{A}' \cap \mathcal{C}_n$.

Remark 3. In many cases (including the examples considered in this paper) one has natural equivalences $D^b(\mathcal{A}') \cong \mathcal{C} \cong D^b(\mathcal{A})$. The resulting equivalence $D^b(\mathcal{A}') \cong D^b(\mathcal{A})$ belongs to a class of equivalences which appeared in the work of Chuang – Rouquier and Craven – Rouquier under the name of *perverse equivalences* see [CR] and references therein (our setting may be slightly more general, but the generalization is straightforward).

Example 2. Let $\mathcal{C} = D^b(\text{Coh}_{\mathcal{B}_e}(X))$ as above, $V = \mathfrak{h}_{\mathbb{R}}^*$ and let Σ consist of the affine coroot hyperplanes. Let V^+ be the positive Weyl chamber. Let $\tau : A \rightarrow \tau_A$ be the map described in [BM, 1.8]. The polynomial map $Z : \mathfrak{h}_{\mathbb{R}}^* \rightarrow K^0(\mathcal{C})_{\mathbb{R}}^*$ is characterized uniquely by its values at the points of the lattice $\Lambda \subset \mathfrak{h}^*$; these values are given by

$$(1) \quad \langle Z(\lambda), [\mathcal{F}] \rangle = \chi(\mathcal{F} \otimes \mathcal{O}(\lambda)),$$

where χ denotes the Euler characteristic and $\mathcal{O}(\lambda)$ is the line bundle attached to λ .

Using Proposition 1 below one can show that this data provides an example of a real variation of stability conditions. Notice that in this case for every pair of neighboring alcoves as in part 2 of the Definition the filtration \mathcal{C}_n is just a two term filtration, i.e. $\mathcal{C}_2 = \{0\}$. Another special feature of this example is that all the t -structures $\tau(A)$ lie in one orbit of the group of automorphisms of \mathcal{C} (in fact, of the group B_{aff} acting on \mathcal{C}).

Remark 4. In a forthcoming work we plan to present more examples of real variations of stability conditions. All our examples are of the following sort: $\mathcal{C} = D^b(\text{Coh}_Z(X))$ where X is a symplectic resolution of singularities and $Z \subset X$ is a closed projective subvariety; $V = N(X) \otimes \mathbb{R}$ where $N(X)$ is the group of numerical classes of divisors on X ; V^+ is spanned by the ample cone, and the central charge map Z is given by a formula similar to (1). The t -structures appearing in the definition of a real variation of stability conditions can in this case be constructed using quantization in positive characteristic, cf. e.g. [BK], [BFG], [K].

Remark 5. Requirement (1) of the Definition implies that $(\sqrt{-1}Z, \tau)$ define a map from V^0 to the space of Bridgeland stabilities $\text{Stab}(\mathcal{C})$. Since V^0 is disconnected, this structure by itself does not provide any relation between the different t -structures, thus it is too weak to yield interesting results. Axiom (2) connects the t -structures assigned to different connected components of V^0 ; it is based on the same intuition as Bridgeland's definition (as we understand it): as x travels from A to A' in the complexification $V_{\mathbb{C}} \setminus H_{\mathbb{C}}$ the phase of a stable objects in $\mathcal{C}_n \setminus \mathcal{C}_{n+1}$ is shifted by $n\pi$, hence the homological shift by n in requirement (2). This heuristics suggests that given a real variation of stability conditions one might expect a map from a connected covering of the complexification $V_{\mathbb{C}}^0 = V_{\mathbb{C}} \setminus \bigcup_{H \in \Sigma} H_{\mathbb{C}}$ to $\text{Stab}(\mathcal{C})$.

The main Theorem of this note is a partial result in that direction. However, the fact that we get a map from a covering of a proper subset in $V_{\mathbb{C}}^0$ which is not even homotopy equivalent to the whole space, and have to use a somewhat unnatural quasi-exponential map is an indication of technical difficulties in connecting the two definitions. We expect even more serious difficulties in the cases when filtrations (\mathcal{C}_n) do not reduce to a two step filtration.

Instead of trying to establish a direct relation between the two structures, it may be more fruitful to view them as different implementations of the same intuition of "physical" origin and possibly try to find a common generalization of the two.

1.5. Real variation of stabilities and automorphisms of derived categories.

In some examples in the literature (see e.g. [Br1], [Br2], [Br3]) (a component of) the space $\text{Stab}(\mathcal{C})$ is realized as a covering of a domain in $K^0(\mathcal{C})_{\mathbb{C}}^*$ where the group of automorphisms of \mathcal{C} acts by deck transformations. We suggest the following counterpart of this picture in the framework of real variations of stability conditions.

Definition 2. A real variation of stability conditions is *symmetric* if the following holds.

- (1) For any alcoves A, A' as in part 2 of Definition 1 there exists an auto-equivalence $m_{A,A'}$ of \mathcal{C} preserving the subcategories $\mathcal{C}_n \subset \mathcal{C}$, so that the induced auto-equivalence of $\mathcal{C}_n/\mathcal{C}_{n+1}$ is isomorphic to the shift functor $M \mapsto M[2n]$.
- (2) The auto-equivalences $m_{A,A'}$ can be chosen so that the following holds. Consider the groupoid $P(V_{\mathbb{C}}^0)$ whose objects are alcoves and morphisms from A to A' are homotopy classes of paths in $V_{\mathbb{C}}^0$ starting at A and ending at A' . Then there exists a functor from $F : P(V_{\mathbb{C}}^0) \rightarrow \text{Cat}$ where Cat is the category of categories with morphisms being the isomorphism classes of functors,¹ such that:
 - (a) $F(A) = \mathcal{C}$ for all $A \in \text{Alc}$.
 - (b) For $A, A' \in \text{Alc}$ as above F sends the class of a path going from A to A' around H in the positive direction to the identity functor.
 - (c) For A, A' as above F sends the class of a path going from A' to A in the positive direction to $m_{A,A'}$.

Remark 6. It is easy to see that a symmetric real variation of stability leads to a more symmetric collection of data, which does not use the choice of the "positive cone" V^+ . Namely, for each alcove A there corresponds a triangulated category \mathcal{C}_A with a t -structure τ_A . To each homotopy class ϖ of a path in $V_{\mathbb{C}}^0$ connecting an alcove A to an alcove A' there corresponds an equivalence $\varphi_{\varpi} : \mathcal{C}_A \rightarrow \mathcal{C}_{A'}$. Moreover, if A and A' share a codimension one face, and ϖ runs from A to A' in the counterclockwise direction, then φ_{ϖ} is compatible with t -structures "up to a shift on the associated graded pieces" as described above.

Having fixed V^+ we can introduce the additional data of an equivalence between \mathcal{C}_A and a fixed triangulated category \mathcal{C} , which is compatible with φ_{π} as above when A' lies above A .

Remark 7. Using the action of B_{aff} it is easy to see that the real variation of stability conditions in Example 2 is symmetric.

More generally, we also expect to obtain symmetric real variation of stability conditions in many cases of the type described in Remark 4. In this situation the heart \mathcal{A}_A of the t -structures τ_A is a characteristic zero lifting of the category of finite dimensional modules for an algebra R_A^k , where k is a field of positive characteristic and R_A^k is the algebra of global sections of a quantization of a symplectic variety over k . The parameter of the quantization should belong to a subset determined by the alcove A . Then $\mathcal{C}_A \cong D^b(\mathcal{A}_A - \text{mod}^{fd})$ by definition, and the equivalence $\mathcal{C}_A \cong \mathcal{C} = D^b(\text{Coh}_{\mathbb{Z}}(X))$ comes from a derived localization theorem in positive characteristic, cf. [BMR1], [B]. Finally, the equivalence $m_{A,A'}$ can be obtained as a composition of the derived localization equivalences and equivalences induced by the isomorphisms $(R_{-A}^k) \cong R_A^k$, which come from an automorphism of X sending the symplectic form ω to $-\omega$.

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¹We present the weak version of the definition for brevity, one can upgrade it to a definition of a finer structure involving the 2-category of categories, or a version of the infinity category of DG -categories.

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2. POSITIVITY PROPERTY

Proposition 1. *Let $A \in \text{Alc}$ and let $M \neq 0$ be an object in the heart of τ_A .*

a) The function $d_M : x \mapsto \langle E(x), [M] \rangle$ is a polynomial taking positive real values on $x \in A$.

b) Let F be a codimension one face of A . Then either d_M takes positive values on F , or $d_M|_F = 0$. In the latter case the object $M[\pm 1]$ lies in the heart of A' where A' is the alcove separated from A by F and the $+$ (respectively, $-$) sign should be taken if A' lies above (respectively, below) A .

Proof. The construction of t -structures τ_A , $A \in \text{Alc}$ on $\mathcal{C} = D^b(\text{Coh}_{\mathcal{B}_e}(X))$ is carried out in [BM] for e, X defined over an algebraically closed field k of arbitrary characteristic, except for positive characteristic $p \leq h$, where h is the Coxeter number of G . Furthermore, for each irreducible object L in the heart of \mathcal{C} , for almost all values of p there exists an object L_k in the heart of τ_A^k with the same class in the Grothendieck group; here we use a standard identification $K^0(\mathcal{B}_{e_c}) = K^0(\mathcal{B}_{e_k})$ for matching nilpotent elements e_c, e_k defined over \mathbb{C} and k respectively, see e.g. [BMR1, §7].

Fix such a prime p , and let $\lambda \in \Lambda$ be such that $\frac{\lambda+\rho}{p} \in A$, where ρ is the sum of fundamental weights. Then it is shown in [BMR1, §6] that $p^{\dim \mathcal{B}} d_M(\frac{\lambda+\rho}{p}) = \dim(\Gamma_\lambda(M))$, where Γ_λ is a certain exact conservative functor (depending on λ) from the heart of τ_A^k to the category of finite dimensional vector spaces over k (in fact, to the category of modules over the Lie algebra \mathfrak{g}_k). Thus for a large prime number p , the polynomial d_M takes positive values at points $\lambda \in A$ such that $p\lambda \in \Lambda$. Since the set of such points, for varying p , is dense, we see that $d_M(\lambda) \geq 0$ for any $\lambda \in A$. It remains to see that the inequality is strict.

Suppose that $d_M(\lambda_0) = 0$ for some $\lambda_0 \in A$. Consider the lowest nonzero term P in the Taylor expansion of the polynomial d_M at λ_0 . This is a homogeneous polynomial on $\mathfrak{h}_{\mathbb{R}}^*$ taking non-negative values only. On the other hand, we claim that d_M , and hence P is a harmonic polynomial, i.e. it is annihilated by any W invariant differential operator with constant coefficients and zero constant term. This follows from the differential equation for the exponential function and the fact that the map $\text{Sym}(\mathfrak{h}^*) \rightarrow H^*(G/B)$ factors through the quotient by the ideal generated by $\text{Sym}(\mathfrak{h}^*)_+^W$. Now, we claim that a harmonic polynomial taking non-negative values only is necessarily zero. Indeed, for a harmonic polynomial P and $\lambda \in \mathfrak{h}^*$ we have $\sum_w P(w(\lambda)) = 0$; if $P(w(\lambda)) \geq 0$, this implies $P \equiv 0$. This proves (a).

The proof of (b) is based on "singular localization" Theorem of [BMR2]. Namely, let $\lambda \in \Lambda$ be such that $\frac{\lambda+\rho}{p}$ lies on a codimension one face of an alcove A . Then again we have $p^{\dim \mathcal{B}} d_M(\frac{\lambda+\rho}{p}) = \dim \Gamma_\lambda(M)$ where the functor Γ_λ is exact but not necessarily conservative (i.e. it may kill some non-zero objects).

Moreover, the functor $b_{A,A'}$ sending τ_A to $\tau_{A'}$ satisfies $b_{A,A'}(L) \cong L[\pm 1]$ for any irreducible object killed by Γ_λ , where the sign is chosen as in the statement of the Lemma. We have $\Gamma_\lambda(M) = 0$ if and only if $\Gamma_\mu(M) = 0$ for every μ with $\frac{\mu+\rho}{p} \in F$. Assuming that p is large enough, we see that if $\Gamma_\lambda(M) = 0$ for some $\frac{\lambda+\rho}{p} \in F$, then the polynomial d_M vanishes at all points of F with sufficiently large prime

denominator, hence $d_M|_F \equiv 0$. In this case we see that $M[\pm 1] \cong b_{A,A'}(M)$ is in the heart of $\tau_{A'}$.

Otherwise if d_M takes positive values at all points of F with a large prime denominator, then an argument involving harmonic polynomials as in the proof of part (a) shows it takes positive values at all points of F . \square

3. PROOF OF THE THEOREM.

We now construct the map as follows. Recall that A_0 denotes the fundamental alcove. It is easy to see that the set

$$S = \{(\lambda, \mu) \mid (\lambda \in A_0) \bigvee (\lambda \in \bar{A}_0, \mu \in A_0)\}$$

is a fundamental domain for the action of W_{aff} on V^{reg} ; it is the intersection of a contractible fundamental domain for the action of W_{aff} on \mathfrak{h}^* with V .

Thus a point in $\widetilde{V^{reg}}$ can be represented by a pair (b, x) where $x \in S$ and b is a homotopy class of path from A_0 to some alcove $A \in Alc$ (the projection to V^{reg} is then given by $(b, x) \mapsto \bar{b}(x)$ where \bar{b} is the element of W_{aff} corresponding to b). We define the map ι by:

$$\iota : (b, x) \mapsto (b(\tau_{A_0}), \sqrt{-1}E(\bar{b}(x))),$$

where we use the same notation for b and the corresponding element of B_{aff} . It remains to check that ι is continuous. This is done using the next

Lemma 2. *Let $\mathcal{F} \in \mathcal{C}$ be an object which is stable with respect to a stability $\sigma \in \widetilde{V^{reg}}$. Then there exists a neighborhood U of σ in $Stab(\mathcal{C})$ such \mathcal{F} is stable with respect to any $\sigma' \in U$.*

Assuming the Lemma we finish the argument as follows. We have to check continuity at the boundary of the region corresponding to a given $b \in B_{aff}$. Without loss of generality we can assume $b = 1$. Let λ be a point in the boundary of S . By Bridgeland openness Theorem [Br] there exists a neighborhood of the point (τ_{A_0}, λ) in $Stab(\mathcal{C})$ mapping isomorphically to a neighborhood of $\sqrt{-1}E(\lambda)$ in $K^0(\mathcal{C})^*$. It suffices to see that for a neighboring alcove $A' = s_\alpha(A_0)$ and a point \tilde{z} in the neighborhood mapping to $z \in s_\alpha(S)$, the t -structure underlying \tilde{z} is $\tau_{A'}$.

Let L be an irreducible object. It suffices to show that $b_{A,A'}(L)$ lies in the heart of the t -structure of \tilde{z} . The argument is similar to [Br1, p. 10]. There are two cases. Either $b_{A,A'}(L) = L[\pm 1]$, then using the Lemma we see that L is stable for \tilde{z} (if the neighborhood is chosen small enough), then $L[\pm 1]$ is in the heart of the t -structure since its phase is in $[0, 1)$. Or $b_{A,A'}(L)$ lies in the heart of τ_A , its Harder-Narasimhan filtration has length two, and both stable subquotients remain stable and have phases in $[0, 1)$ in stability \tilde{z} . \square

It remains to prove Lemma 2. The proof below was suggested to us by Tom Bridgeland.

Proof. It is easy to see that any $\sigma \in \iota(\widetilde{V^{reg}})$ is locally finite. If \mathcal{F} is stable in $\sigma = (P, Z)$ of phase t then using local-finiteness we can find a finite length quasi-abelian category $\mathcal{B} = P(t - a, t + a)$ (notations of [Br]). The object \mathcal{F} has a finite Jordan-Hoelder series in \mathcal{B} . Thus there are only finitely many elements in $K^0(\mathcal{C})$ which can be represented by a subobject of \mathcal{F} in \mathcal{B} , since such a class is a sum of classes of some of the simple subquotients. Since \mathcal{F} is stable, the class of a subobject $\mathcal{F}' \neq \mathcal{F}$ has phase strictly less than t . Thus there exists $s < t$ such that the class of

any such subobject has phase less than s . If U is a sufficiently small neighborhood of σ , then the phase of any subobject of \mathcal{F} with respect to $\sigma' \in U$ differs from its phase with respect to σ by less than $\frac{t-s}{2}$. Thus the phase of subobject $\mathcal{F}' \neq \mathcal{F}$ with respect to σ' is less than the phase of \mathcal{F} with respect to σ' , which shows that \mathcal{F} is stable with respect to σ' . \square

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